Big + Pseudoeffective cones

The big cone $\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the cone of big $\mathbb{R}$-divisors.

The pseudoeffective cone $\overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the numerical classes of effective $\mathbb{R}$-divisors. $[D] \in \overline{\mathrm{Eff}}(X)$ is effective if it has an $\mathbb{R}$-effective representative.

Rmk: If $[D] \in \mathrm{Big}(X)$, then D is big.

However, $D \equiv_{num} E$ effective $\not\Rightarrow D$ effective (even up to lin. equiv)

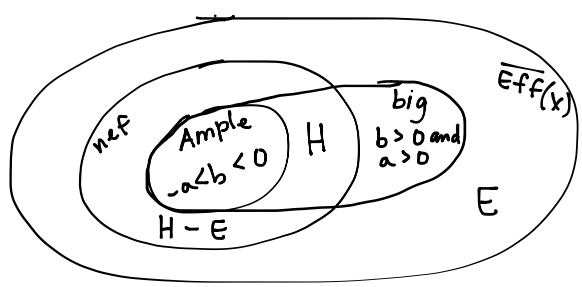
Ex: Let C be a curve and D a nontorsion divisor of deg 0. Then D is not effective in $\mathrm{Div}_{\mathbb{R}}(X)$.

However, $D \equiv_{num} 0$ so $[D] = [0] \in N^1(X)_{\mathbb{R}}$

Exercise: Come up w/ a higher dim'l example.

Example: Let $X = Bl_p \mathbb{P}^2$, H = pullback of hyp, E = exceptional

$$\overline{\mathrm{Eff}}(X) = \overline{NE}(X) = \{aH + bE \mid a \geq 0,\ a \geq -b\}$$



Apr 3

$\underline{\text{Thm}}$: $\mathrm{Big}(X) = \mathrm{int}(\overline{\mathrm{Eff}}(X))$ and $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}(X)}$

$\underline{\text{Pf}}$: We know big = ample + eff, and ample is effective, so

$\mathrm{Big}(X) \subseteq \overline{\mathrm{Eff}}(X)$

Also, $\overline{\mathrm{Eff}}(X)$ is closed and $\mathrm{Big}(X)$ is open, so

$\mathrm{Big}(X) \subseteq \mathrm{int}(\overline{\mathrm{Eff}}(X))$ and $\overline{\mathrm{Big}(X)} \subseteq \overline{\mathrm{Eff}}(X)$

Just need the reverse inclusions.

Assume $D \in \mathrm{int}(\overline{\mathrm{Eff}}(X))$. Fix A ample.

Then $D + \varepsilon(-A) \in \mathrm{int}(\overline{\mathrm{Eff}}(X))$ for $0 < \varepsilon << 1$.

$\Rightarrow$ $D - \varepsilon A \equiv_{num} N$ for some effective class N (since $N \in$ interior)

$\Rightarrow$ $D \equiv_{num}$ ample + effective $\Rightarrow$ D is big.

Now assume $D \in \overline{\mathrm{Eff}}(X)$. Then

$D = \lim_{k \to \infty} D_k$ where the D_k's are effective.

Fix A ample. Then $\underbrace{D_k + \frac{1}{k}A}_{\text{Big}} \to D$,

so D is a limit of big divisors $\Rightarrow$ $D \in \overline{\mathrm{Big}(X)}$. $\square$